An Addition Table with Interesting Properties

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Introduction

Here is a normal addition table:

0	1	2	3	4	5	6	7
1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10
4	5	6	7	8	9	10	11
5	6	7	8	9	10	11	12
6	7	8	9	10	11	12	13
7	8	9	10	11	12	13	14

Let's pretend it goes on to infinity. **Is it possible to delete certain rows and columns so that each nonnegative integer appears exactly once?** Yes. We can delete all the rows except the first, or all the columns except the first, and each nonnegative integer appears exactly once.

0	1	2	3	4	5	6	7	⋯

or

0
1
2
3
4
5
6
7
⋮

But these solutions are trivial. Is there a way to do it that leaves infinitely many rows and infinitely many columns?

Solving the Puzzle

Let's solve this problem one nonnegative integer at a time. We need a 0 in the final table, so we can't delete the first row or column. There are two 1's in the table. One of them must be deleted, so let's delete the 1 in the first column. We can't delete it by deleting the first column, so we have to delete the second row:

0	1	2	3	4	5	6	7
●	●	●	●	●	●	●	●
2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10
4	5	6	7	8	9	10	11
5	6	7	8	9	10	11	12
6	7	8	9	10	11	12	13
7	8	9	10	11	12	13	14

Now there are two 2's. We can delete the third column or the third row. Let's choose the third column:

0	1	●	3	4	5	6	7
●	●	●	●	●	●	●	●
2	3	●	5	6	7	8	9
3	4	●	6	7	8	9	10
4	5	●	7	8	9	10	11
5	6	●	8	9	10	11	12
6	7	●	9	10	11	12	13
7	8	●	10	11	12	13	14

We can't delete the 3 in the third row and the second column because deleting the third row would delete the only 2 left, and deleting the second column would delete the only 1 left. So this 3 must be in the final table, and we have to delete the other two 3's, which means deleting the fourth row and third column:

0	1	●	●	4	5	6	7
●	●	●	●	●	●	●	●
2	3	●	●	6	7	8	9
●	●	●	●	●	●	●	●
4	5	●	●	8	9	10	11
5	6	●	●	9	10	11	12
6	7	●	●	10	11	12	13
7	8	●	●	11	12	13	14

Notice that the numbers 0, 1, 2, and 3 are in the top-left 2-by-2 square, and 4, 5, 6, and 7 are in the square to its right. Let's keep that square for the final table, and delete all the other 4's, 5's, 6's, and 7's. There are a lot of red dots now, so we'll delete them and add in an extra row:

0	1	4	5
2	3	6	7
8	9	12	13

We see a partial 8, 9, 10, 11 square on the bottom edge. We also see a partial 12, 13, 14, 15 square. Notice that the four 2-by-2 squares are in a 2-by-2 square in the same order (top-left, top-right, bottom-left, bottom-right)!

Look at the final table (in the next block of the poster). The 2-by-2 squares 0-3, 4-7, 8-11, and 12-15 are arranged in the 4-by-4 square 0-15 in the same pattern as 0, 1, 2, and 3 in the 2-by-2 square 0-3. The 4-by-4 squares 0-15, 16-31, 32-47, and 48-63 are arranged in the 8-by-8 square 0-63 in the same pattern.

Note: This is not the only addition table with infinitely many rows and columns such that every nonnegative integer appears exactly once. We will call the property that every nonnegative integer appears exactly once "**Property 1**."

The Final Table (Top-Left 16 by 16)

0	1	4	5	16	17	20	21	64	65	68	69	80	81	84	85
2	3	6	7	18	19	22	23	66	67	70	71	82	83	86	87
8	9	12	13	24	25	28	29	72	73	76	77	88	89	92	93
10	11	14	15	26	27	30	31	74	75	78	79	90	91	94	95
32	33	36	37	48	49	52	53	96	97	100	101	112	113	116	117
34	35	38	39	50	51	54	55	98	99	102	103	114	115	118	119
40	41	44	45	56	57	60	61	104	105	108	109	120	121	124	125
42	43	46	47	58	59	62	63	106	107	110	111	122	123	126	127
128	129	132	133	144	145	148	149	192	193	196	197	208	209	212	213
130	131	134	135	146	147	150	151	194	195	198	199	210	211	214	215
136	137	140	141	152	153	156	157	200	201	204	205	216	217	220	221
138	139	142	143	154	155	158	159	202	203	206	207	218	219	222	223
160	161	164	165	176	177	180	181	224	225	228	229	240	241	244	245
162	163	166	167	178	179	182	183	226	227	230	231	242	243	246	247
168	169	172	173	184	185	188	189	232	233	236	237	248	249	252	253
170	171	174	175	186	187	190	191	234	235	238	239	250	251	254	255

Figure 1: The final table

Properties of the Table

Let a_n be the first row sequence of the table (so $a_0 = 0$, $a_1 = 1$, $a_2 = 4$, $a_3 = 5$, and so on), and let b_n be the first column sequence. Since the table is an addition table, the entry in row i and column j is $b_{i-1} + a_{j-1}$. Here are a few properties of a_n and b_n (all the variables are nonnegative integers):

- $b_n = 2a_n$
- $a_{2n} = 4a_n$
- $a_{2n+1} = 4a_n + 1$
- $a_{2^n} = 4^n$
- $a_{2^n-1} = (4^n - 1)/3$
- If $n = 2^k q + r$ where $r < 2^k$ then $a_n = 4^k a_q + a_r$. (Most of the properties above this one are special cases of this one.)
- All nonnegative integers n can be uniquely expressed as $2a_i + a_j$ for some nonnegative integers i and j. (This proves that every number appears exactly once in the table.)
- The differences of a, Δa_n, are equal to
$$\Delta a_n = a_{n+1} - a_n = \frac{2 \cdot 4^{E_2(n+1)} + 1}{3},$$
where $E_2(n+1)$ is the exponent of 2 in the prime factorization of $n+1$.
- The preceding property implies that
$$a_n = \frac{1}{3}\left(n + 2\sum_{k=1}^{n} 4^{E_2(k)}\right).$$
- The sum of a_k from $k = 0$ to $2^n - 1$ is
$$\sum_{k=0}^{2^n-1} a_k = \frac{8^n - 2^n}{6} = \binom{2^n+1}{3}.$$

How to Calculate the First Row

How can we calculate a_n? One way is to use the formula given in the previous block. Here is an example:

$$a_5 = \frac{1}{3}\left(5 + 2\sum_{k=1}^{5} 4^{E_2(k)}\right) = \frac{1}{3}\left(5 + 2\left(4^0 + 4^1 + 4^0 + 4^2 + 4^0\right)\right) = \frac{1}{3}(5 + 2 \cdot 23) = 17,$$

which is correct. But this involves calculating $E_2(k)$ for all k between 1 and n. Do we have to do this, or is there a faster way to calculate a_n?

There is a faster way, and if we wrote numbers in binary instead of decimal, it would have been obvious. Here is the sequence a_n written in binary:

n (binary)	0	1	10	11	100	101	110	111	1000	1001	1010	1011	⋯
a_n (binary)	0	1	100	101	10000	10001	10100	10101	1000000	1000001	1000100	1000101	⋯

The pattern is even more obvious if we look at n in binary and a_n in quaternary (base 4):

n (binary)	0	1	10	11	100	101	110	111	1000	1001	1010	1011	⋯
a_n (quaternary)	0	1	10	11	100	101	110	111	1000	1001	1010	1011	⋯

We see that to calculate a_n, we can find the digits of n in binary and interpret them as quaternary digits. Most of the properties listed in the previous block can be proved pretty easily using this fact. Also, we see that a number is equal to a_n for some n if and only if it only has 0's and 1's in its quaternary representation. This method of calculating a_n is very easy for computers to do because computers store numbers in binary. The binary digit of n with place value 2^k is

$$d_k(n) = \left\lfloor \frac{n}{2^k} \right\rfloor \bmod 2,$$

so we have another formula for a_n:

$$\textbf{(Formula 1)} \quad a_n = \sum_k 4^k d_k(n) = \sum_k 4^k \left(\left\lfloor \frac{n}{2^k} \right\rfloor \bmod 2\right).$$

(Note: Since $d_k(n) = 0$ when $k < 0$ or $k > \lfloor \log_2 n \rfloor$, we are only summing a finite number of terms.)

Other Addition Tables

Are there other addition tables that have Property 1?

- The reflection about the main diagonal of any table with Property 1 also has Property 1.
- For each integer $c \geq 2$, define
$$a_n^{(\text{base } c)} = \sum_k c^{2k}\left(\left\lfloor \frac{n}{c^k} \right\rfloor \bmod c\right).$$
The table with (i, j)th entry $ca_i^{(\text{base } c)} + a_j^{(\text{base } c)}$ has Property 1. A lot of properties listed in the "Properties of the Table" block can be extended to $a^{(\text{base } c)}$. (When $c = 2$ this becomes the table in Figure 1. The number $a_n^{(\text{base } c)}$ is the digits of n in base c interpreted as digits in base c^2.)
- There are more than just these.

The table in Figure 1 is a two-dimensional array. What about higher-dimensional addition tables? For each pair of integers (c, d) both at least 2, define

$$\textbf{(Formula 2)} \quad a_n^{(\text{base } c, \text{ dim } d)} = \sum_k c^{dk}\left(\left\lfloor \frac{n}{c^k} \right\rfloor \bmod c\right).$$

The d-dimensional table with $\mathbf{n} = (n_1, n_2, \ldots, n_d)$-th entry

$$\sum_{j=1}^{d} c^{d-j} a_{n_j}^{(\text{base } c, \text{ dim } d)}$$

has the d-dimensional version of Property 1. (The number $a_n^{(\text{base } c, \text{ dim } d)}$ is the digits of n in base c interpreted as digits in base c^d.)

The Addition Table in $\mathbb{R}^2$

In this block, $\mathbb{R}_+$ means the set of *nonnegative* real numbers. Let $g : \mathbb{R}_+^d \to \mathbb{R}_+$. We call g a *real number addition table in d dimensions* if for all $\mathbf{x} \in \mathbb{R}_+^d$,

$$g(\mathbf{x}) = \sum_{j=1}^{d} g(C_j \mathbf{x}),$$

where C_j is the matrix with (j, j)th entry 1 and all the other entries 0. We can extend Property 1 to real number addition tables by saying g has Property 1 if g is bijective.

Can we define a_n if n is any real number? Yes. We can use Formula 1 from the "How to Calculate the First Row" block. Since this is a function from $\mathbb{R}$ to $\mathbb{R}$, we will call it $f(x)$ instead of a_n. Here are some properties of f:

- The sum in Formula 1 converges for all real numbers.
- Every nonnegative real number can be written uniquely as $2f(x) + f(y)$ for some x and y. Therefore, the function $g(x, y) = 2f(x) + f(y)$ is a real number addition table in two dimensions.
- Let q be a nonnegative integer, k be a (possibly negative) integer, and r be a nonnegative real number less than 2^k. Then, $f(2^k q + r) = 4^k f(q) + f(r)$.
- The range of f has measure 0 but is uncountable. It is very similar to the Cantor set.
- The function f is continuous at a real number $x > 0$ if and only if $2^k x \notin \mathbb{N}$ for any integer k. (So f is continuous at all nonterminating binary numbers, and discontinuous at all terminating binary numbers.)
- Since the set of discontinuities of f has measure 0, we can integrate f. Let $n \in \mathbb{Z}$. The integral of f from 0 to 2^n is
$$\int_0^{2^n} f(x)\,dx = \frac{8^n}{6}.$$
- The integral of f from 0 to a rational number of the form $2^t/(2^v - 1)$ where $t \in \mathbb{Z}$ and $v \in \mathbb{N}$ is
$$\int_0^{2^t/(2^v-1)} f(x)\,dx = \frac{8^t}{8^v - 1}\left(\frac{1}{2^v - 1} + \frac{1}{6}\right).$$
If $v = 1$ then this reduces to the preceding property.
- Let $n \in \mathbb{N}$. If we know the integral of f from 0 to n, then we can find the integral of f from 0 to a rational number of the form $2^t n/(2^v - 1)$, where $t \in \mathbb{Z}$ and $v \in \mathbb{N}$:
$$\int_0^{2^t n/(2^v-1)} f(x)\,dx = \frac{8^t}{8^v - 1}\left(\frac{1}{2^v - 1} + \int_0^n f(x)\,dx\right).$$
If $n = 1$ then this reduces to the preceding property.
- The third property in this list implies that
$$\int_{2^k q}^{2^k q + r} f(x)\,dx = 4^k r f(q) + \int_0^r f(x)\,dx,$$
where q is a nonnegative integer, $k \in \mathbb{Z}$, and r is a nonnegative real number less than 2^k.
- For all nonnegative integers n,
$$\int_0^n f(x)\,dx = \frac{n}{6} + \sum_{i=0}^{n-1} f(i).$$
- For all $x \in \mathbb{R}_+$ and $k \in \mathbb{Z}$,
$$f(x + 2^k) - f(x) = 4^k \cdot \frac{2 \cdot 4^{E_2(\lfloor 2^{-k} x \rfloor + 1)} + 1}{3}.$$
- Let $x = n/2^k$ be a point at which f is discontinuous, where n is odd and $k \in \mathbb{Z}$. Then, the right limit of f at x is $f(x)$ and the left limit is $f(x) - 2 \cdot 4^{-k}/3$.
- We can use Formula 2 from the "Other Addition Tables" block to define $f^{(\text{base } c, \text{ dim } d)}(x)$ to get infinitely many real number addition tables in each dimension which have Property 1.