Binomial coefficients linked with Sierpiński & Riesel numbers

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Introduction

A **Sierpiński number** is an odd integer k such that $k \cdot 2^n + 1$ is composite for all $n \in \mathbb{N}$. A **Riesel number** is an odd integer k such that $k \cdot 2^n - 1$ is composite for all $n \in \mathbb{N}$. A **covering system** is a set of congruences $x \equiv a_i \pmod{n_i}$ such that all integers satisfy at least one of the congruences.

Filaseta, Finch, and Kozek asked the following: for a polynomial $f(k)$, is there a k such that $f(k)$ is a Sierpinski number?

In 2013, Finch, Harrington, and Jones proved the following theorem.

Theorem. Let $f(x) = x^r + x + c \in \mathbb{Z}[x]$, where $0 \le c \le 100$.

- (Nonlinear Sierpiński Numbers) For any positive integer r and any $c \in C_1$ there exist infinitely many positive integers k such that $f(k) \cdot 2^n + 1$ is composite for all integers $n \ge 1$.
- (Nonlinear Riesel Numbers) For any positive integer r, and any $c \in C_2$, there exist infinitely many positive integers k such that $f(k) \cdot 2^n - 1$ is composite for all integers $n \ge 1$.

Binomial Coefficients and Sierpiński numbers

Lemma. Let $p = 641$, and let

$\mathscr{G} = \{\gamma \in [1,p-1] : \gamma \text{ is odd}\} \cup$

$\{2, 6, 8, 10, 12, 22, 24, 30, 32, 34, 44, 46, 48, 52, 56, 66, 70, 74, 80, 84, 86, 94, 100, 102, 104, 110, 118, 120, 134, 136, 140, 144, 146, 160, 162, 174, 176, 182, 184, 190, 194, 198, 200, 202, 208, 222, 224, 236, 248, 250, 252, 260, 270, 292, 294, 304, 312, 318, 334, 336, 338, 348, 366, 368, 374, 402, 414, 424, 426, 454, 474, 530, 546, 552, 578\}.$

Then there exists a function $\kappa : \mathscr{G} \to [0,p-1]$ such that for every $r \in \mathscr{G}$, $\binom{\kappa(r)}{r} \equiv -1 \pmod{p}$.

Theorem 1. Let $p = 641$, and recall $\mathscr{G}$ defined in the Lemma. Let r be a nonnegative integer with base p representation $r = \sum_{i=0}^{j} r_i p^i$, where $r_i \in [0,p-1]$ for all $i \in [0,j]$, such that at least one of the following conditions is satisfied:

- there exists $i_0 \in [0,j]$ such that $r_{i_0} \in \mathscr{G}$; or
- there exists $i_1, i_2 \in [0,j]$ such that $r_{i_1}, r_{i_2} \in [1,515]$.

Then there exist infinitely many positive integers k such that $\binom{k}{r}$ is a Sierpiński number.

Corollary. Let r be an odd positive integer. Then there exist infinitely many positive integers k such that $\binom{k}{r}$ is a Sierpiński number.

Generalizations of Sierpiński and Riesel Binomial Coefficients

For a positive integer a, we call a positive integer k an a-Sierpiński (resp. a-Riesel) number if $\gcd(k+1, a-1) = 1$ (resp. $\gcd(k-1, a-1) = 1$), k is not a power of a, and $k \cdot a^n + 1$ (resp. $k \cdot a^n - 1$) is composite for all natural numbers n.

The following theorem extends the corollary to a-Sierpiński and a-Riesel numbers.

Theorem 2. Let a and r be positive integers such that $a+1$ is not a power of 2 and r is odd. Further assume that there exists a positive integer τ such that $a^{2^\tau} - 1$ is divisible by distinct primes p_0 and $p_{\tau'}$, where neither p_0 nor p_τ divides $a^{2^{\widetilde{\ell}}} - 1$ for any $\widetilde{\ell} \in [0, \tau-1]$. Then each of the following holds:

- there exist infinitely many positive integers k such that $\binom{k}{r}$ is an a-Sierpiński number;
- there exist infinitely many positive integers k such that $\binom{k}{r}$ is an a-Riesel number.

Generalizations using (a,b)-primitive m-coverings

Harrington extended the concept of $(2,1)$-*primitve* $m - coverings$ in 2015 with the following definition: A covering system $\mathscr{C} = \{q_\ell \pmod{m_\ell}\}_{\ell=1}^{\tau}$ is called an (a,b)-*primitive* m-*covering* if every integer satisfies at least m congruences of $\mathscr{C}$ and there exist distinct primes $p_1, p_2, \ldots, p_\tau$ such that for each $\ell \in [1,\tau]$, $p_\ell \mid a^{m_\ell} - b^{m_\ell}$ and $p_\ell \nmid a^{\widetilde{\ell}} - b^{\widetilde{\ell}}$ for any $\widetilde{\ell} < m_\ell$. It is a (a,b)-*primitive* disjoint m-*covering* if it can be partitioned into m disjoint (a,b)-primitive 1-covering systems.

Theorem 3. Let a be a positive integer for which there exists an $(a,1)$-primitive m-covering $\mathscr{C}$. Then there exist infinitely many positive integers r for which each of the following holds:

- there exist infinitely many positive integers k such that $\gcd\left(\binom{k}{r}+1, a-1\right) = 1$, $\binom{k}{r}$ is not a power of a, and $\binom{k}{r} \cdot a^n + 1$ has at least m distinct prime divisors for all natural numbers n;
- there exist infinitely many positive integers k such that $\gcd\left(\binom{k}{r}-1, a-1\right) = 1$, $\binom{k}{r}$ is not a power of a, and $\binom{k}{r} \cdot a^n - 1$ has at least m distinct prime divisors for all natural numbers n; and
- if $\mathscr{C}$ is an $(a,1)$-primitive disjoint m-covering, then there exist infinitely many positive integers k such that $\gcd\left(\binom{k}{r}+1, a-1\right) = \gcd\left(\binom{k}{r}-1, a-1\right) = 1$, $\binom{k}{r}$ is not a power of a, $\binom{k}{r} \cdot a^n + 1$ and $\binom{k}{r} \cdot a^n - 1$ are composite, and each of $\binom{k}{r} \cdot a^n + 1$ and $\binom{k}{r} \cdot a^n - 1$ has at least $\lfloor m/2 \rfloor$ distinct prime divisors for all natural numbers n.

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